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Squiggle Orbs: Segmented-torus Truchet Tilings of the Sphere

Matt Zucker

Swarthmore College, Swarthmore, PA USA; mzucker@swarthmore.edu

Abstract

Truchet tiles produce complex and visually appealing patterns, and have been extended to an ever-growing variety of lattices. In this work, I present a generalization of Truchet tiles to the uniform spherical polyhedra that uses segments of tori to decorate tiles. I also describe the computational design and fabrication of modular, 3D printed spherical Truchet tiles which can be freely repositioned on an underlying spherical lattice.

Introduction

More than three centuries ago, Sébastien Truchet discovered a simple, fixed set of square tiles that could fill the plane with varied and interesting patterns [5]. Truchet's idea, shown in Figure 1, was re-popularized in the 1980s by Cyril Stanley Smith [4], who proposed the variant shown in Figure 2. In subsequent years, researchers and artists have proposed numerous variations on the theme of Truchet tiling beyond the square grid, including hyperbolic tilings and adaptations to Archimedian tilings of the plane [2, 3]. In a similar vein, this paper presents an adaptation of Truchet tiles to uniform spherical polyhedra that incorporates arc-shaped segments of tori, as illustrated in Figure 3. Due to the serpentine paths on the surface of the sphere created by the tilings, I have named this family of tilings "squiggle orbs".

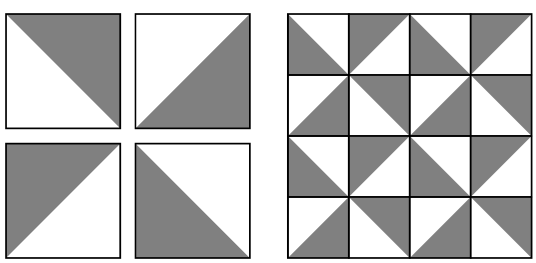

Figure 1: *Truchet's original tiles (left) and an example tiling (right). Reproduced from [3].*

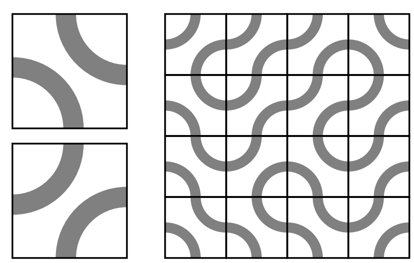

Figure 2: *Smith's tiles (left) and an example tiling (right). Reproduced from [3].*

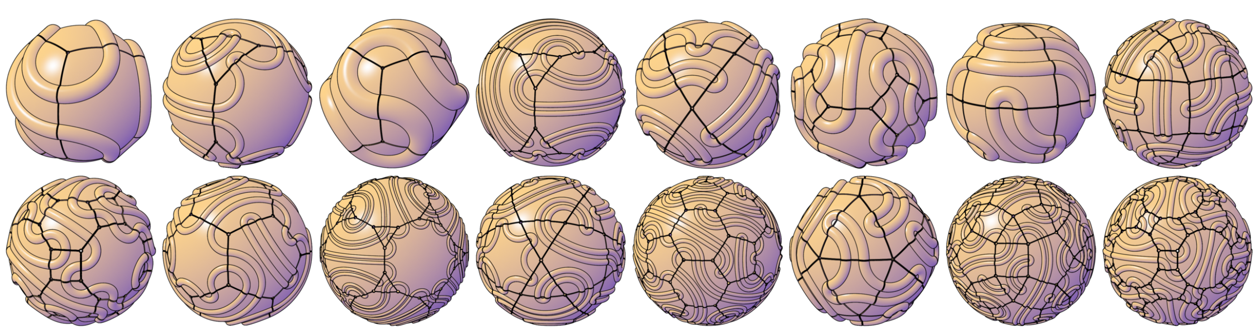

Figure 3: *Segmented-torus Truchet tilings of the sphere. Each tile is embossed with multiple torus segments. These are all of the uniform spherical polyhedra except for the dihedral and snub tilings.*

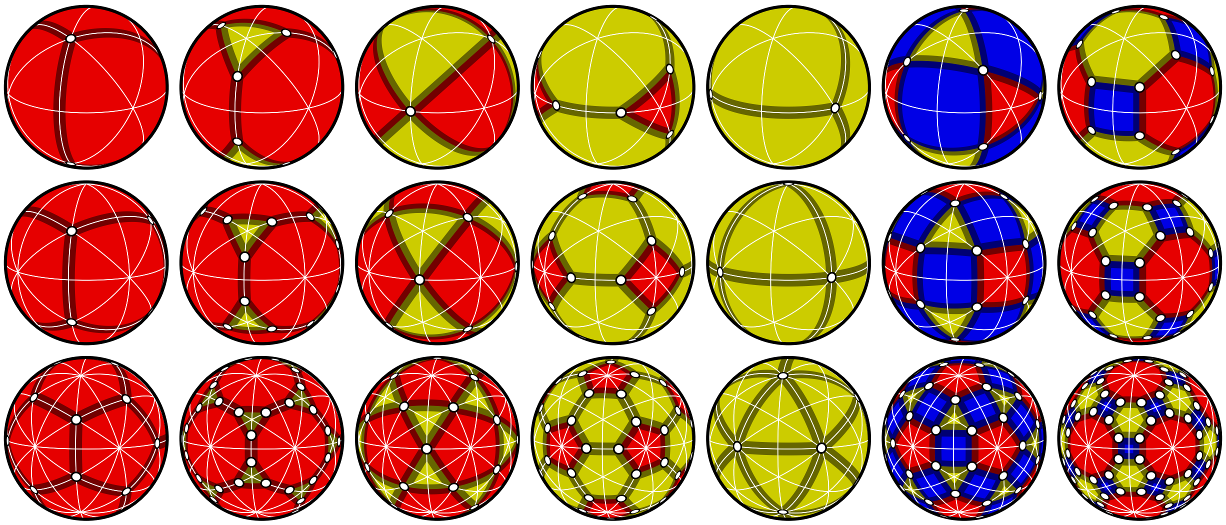

Figure 4: *Wythoff's kaleidoscopic construction of spherical tilings. Each row corresponds to a specific lattice (white lines) and each column to a specific vertex position (white spot) within the unit cell.*

Geometric Construction of Squiggle Orbs

The uniform spherical polyhedra can be constructed using Wythoff's kaleidoscopic method [6]. As illustrated in Figure 4, it begins by establishing a triangular lattice on the sphere with interior angles $\pi/2$, $\pi/3$, and π/p where $p \in \{3,4,5\}$. Next, a polygon vertex is positioned at one of several key locations—triangle vertex, edge, or incenter—within the repeating unit cell. Then each polygon edge in the tiling is formed by a mirrored pair of vertices in adjacent cells. Since each spherical polygon is centered on some lattice node, the underlying lattice forms a useful modular scaffold to which tiles can be affixed. Magnets are placed at each node on a physical model of the lattice, designed to mate with magnets placed at the center of each polygonal tile. Then, any specific lattice can be reused to build all of the tilings in its respective row of Figure 4.

Like Smith and Mitchell [4, 3], I decorate every n-sided polygonal tile with either n or $n/2$ arc segments. In the case of $n/2$ arc segments, the segments all intersect the polygon edges at their midpoints (e.g., as in the bottom-left tiling of Figure 3), and for tiles with n arc segments, the segments intersect the polygon edges at pairs of points equally spaced from the midpoints (e.g., as in the top-right tiling of Figure 3). Regardless of the number of arcs, the arc segments always intersect polygon edges at right angles, ensuring G1 continuity across polygon edges. As a result of these properties, any tile can be rotated in-place or swapped with another one of the same shape without "breaking the pattern".

In a spherical polyhedron, polygon edges are segments of great circles, and arcs correspond to segments of small circles, as illustrated in Figure 5. Consider the great circles ℓ_1 and ℓ_2 on the sphere containing two polygon edges which we wish to connect with an arc. Then the projection of the arc center onto the sphere lies at an intersection point of the two great circles, defined as $\mathbf{p}_C = (\ell_1 \times \ell_2)/\|\ell_1 \times \ell_2\|$. Once we choose a point $\mathbf{p}_1$ along line ℓ_1 to begin the arc segment, the end of the arc segment is fixed at the point $\mathbf{p}_2$ along ℓ_2 which lies at the same distance from $\mathbf{p}_C$ as $\mathbf{p}_1$. When ℓ_1 and ℓ_2 are coincident, then $\mathbf{p}_C$ is the edge midpoint.

Rather than simply inscribing them with circular arcs, the tiles are embossed with segments of tori formed by sweeping a sphere along each arc segment. The axis of rotational symmetry of the torus is the unit vector $\mathbf{p}_C$. The center of the torus can be defined as $\mathbf{t}_C = (\mathbf{p}_C \cdot \mathbf{p}_1)\mathbf{p}_C$, and the major radius of the torus is defined as $R = \|\mathbf{t}_C - \mathbf{p}_1\|$. The minor radius r is a free parameter, and I choose it to be a certain fraction of the polygon edge length. In order to accommodate two arcs per edge, tori in tiles with n arcs are given a smaller minor radius than those in tiles with $n/2$ arc segments.

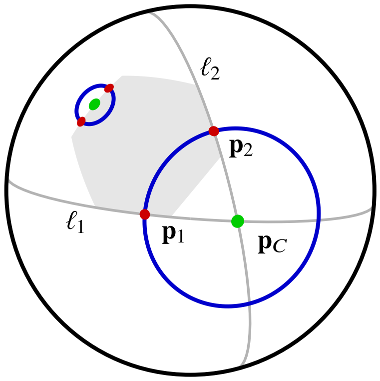

Figure 5: *Constructing arcs to connect pairs of points on polygon edges.*

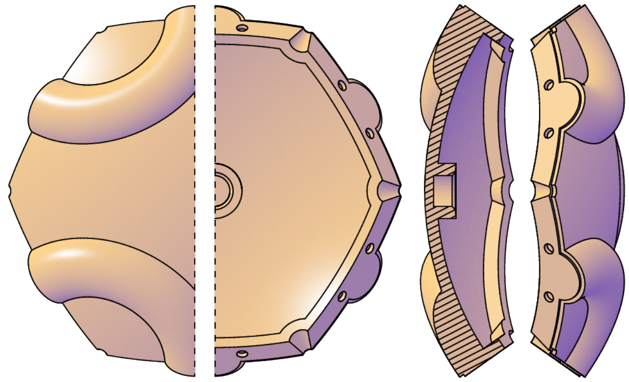

Figure 6: *Front, back, cross-sectional, and side views of a hexagonal tile.*

Computational Design

I wrote a Python program using the CadQuery kernel [1] to design printable 3D models of the scaffolds and tiles, which are both directly output by the program in STL format. The tiles incorporate several key design features, detailed in Figure 6. The embossed torus segments on the front of the tile create distinctive patterns of light and shadow across the sphere. On the back of the tile, there is a circular pocket for housing a disc magnet. A small chamfer aids in positioning the magnet during assembly. The side of the tile has multiple notable features, starting with a stepped "reveal" which creates an apparent gap between tiles, and which helps to hide small variations in tile spacing due to positioning or printing inaccuracies. Circular registration marks along the edge indicate tile compatibility – any given tiling will consist of tiles whose registration marks are equal in number and in spacing. Conical cutouts at the polygon vertices serve two purposes: like the stepped "reveal", they help hide small errors in tile alignment at the corners; additionally, their conical shape allows insertion of a toothpick to pry tiles away from the sphere during disassembly.

The scaffolds, illustrated in Figure 7, are formed from hollow spheres with triangular cutouts. Each node on the lattice has a chamfered circular pocket designed to house a magnet. When gluing magnets into the scaffolds and tiles, it is essential to ensure that they are installed with compatible polarity. That is, every disc magnet on the scaffold should have its north pole facing outwards, and every magnet on a tile interior should have its south pole facing inwards (or vice versa). Although the tiles and scaffolds could be printed at any size, I chose sizes based on affordability and ease of manipulation and transport. Each scaffold measures approximately 90 mm, and finished tilings are 105-115 mm in diameter. Tiles range in size from $13 \times 13 \times 7$ mm to $90 \times 90 \times 34$ mm. See Figure 8 for example 3D printed tilings.

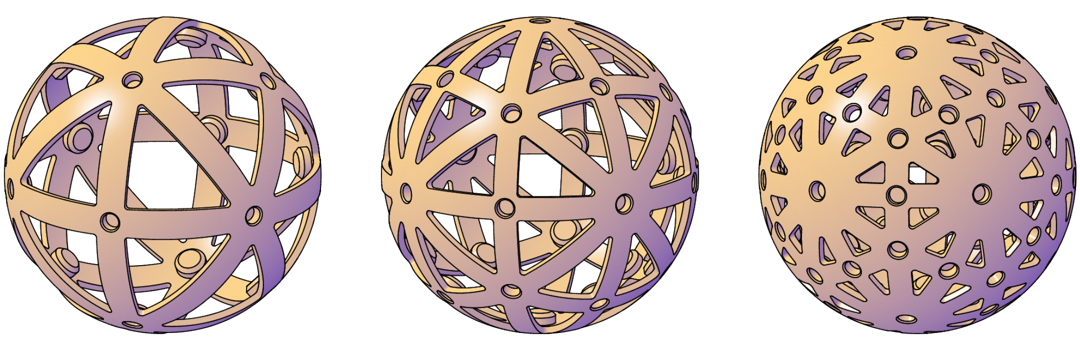

Figure 7: *Scaffolds to support tilings, each corresponding to the lattice in one row of Figure 4.*

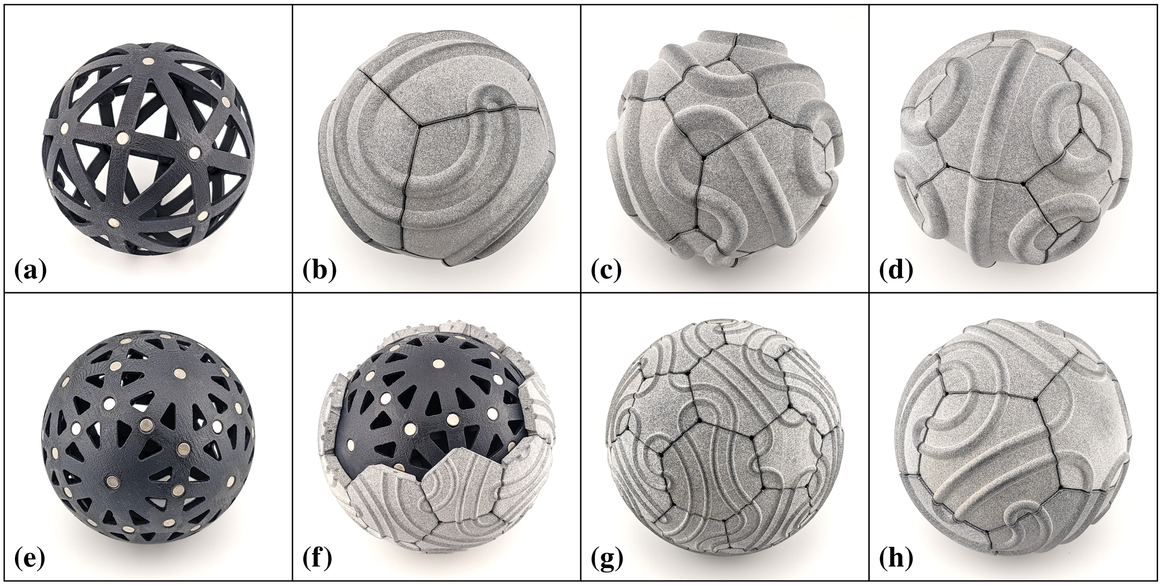

Figure 8: *3D printed objects, all ~100 mm diameter: (a) octahedral lattice; (b) 4^3 tiling; (c-d) two distinct 4.6.6 tilings that minimize/maximize disjoint path count; (e) icosahedral lattice; (f-g) partial and full 5.6.6 tilings; (h) 5^3 tiling. Lattices are SLS-printed nylon and tiles are MJF-printed nylon.*

Classroom Activities and Future Work

Squiggle orbs are aesthetically pleasing and make excellent desk/fidget toys. Furthermore, they are vehicles for entertaining mathematical inquiry at multiple education levels. At the primary school level, students could be asked to state the rule that governs tiles can be placed on the lattice to cover the sphere. (*Solution: lattice lines must perpendicularly bisect tile edges.*) At the middle or high school level, students could be challenged to rearrange tiles to maximize or minimize the number of disjoint paths made from the torus segments. And at the college or graduate level, students could be asked to analyze how many distinct patterns could be assembled from a given tile set, up to symmetries. Near-term future work includes experimenting with materials and surface finishes to improve upon the squiggle orbs I have already printed. In the longer term, I hope to investigate space-filling polyhedral "Truchet building blocks".

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