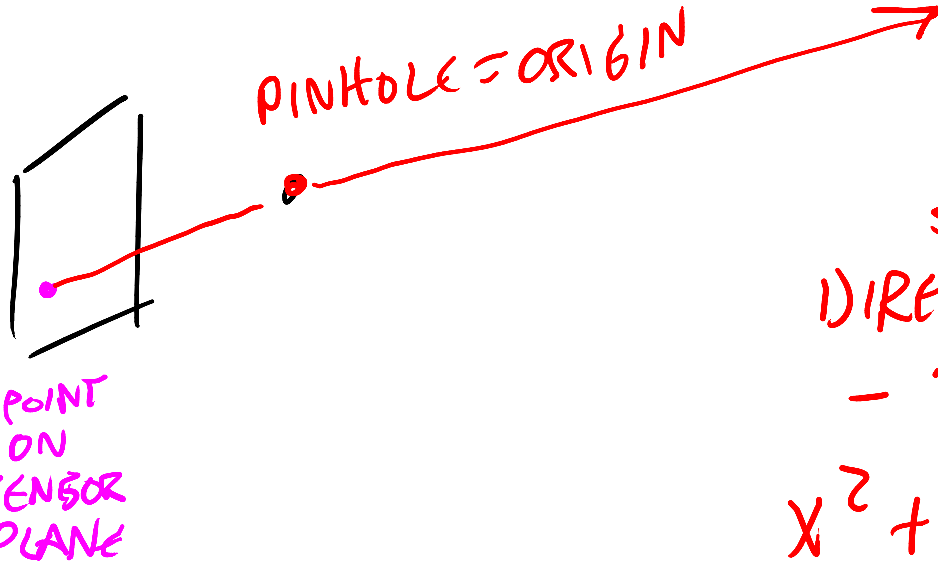
TODAY
- INTRO TO MULTIPLE VIEW GEOMETRY
- POINTS & LINES IN 2D

BIG IDEA! EQUATE POINTS IN 2D WITH RAYS IN 3D

→ HOMOGENEOUS COORDINATES



2 DOF (x, y)

DIRECTION IS UNIT VECTOR

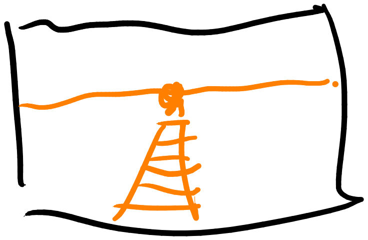
- 2 DEGREES OF FREEDOM (DOF)

$$x^2 + y^2 + z^2 = 1$$

$$z = \sqrt{1 - x^2 - y^2}$$

WHY HOMOGENEOUS COORDS:

- REPRESENT TRANSFORMS AS LINEAR
- REPRESENT POINTS AND LINES IN THE SAME FRAMEWORK
- REASON ABOUT IDEAL POINTS AKA POINTS AT INFINITY



POINTS AT INFINITY $\Longleftrightarrow$ VANISHING POINTS IN IMAGES

TRADITIONAL CARTESIAN COORDINATES

$$p \in \mathbb{R}^2 \qquad p = \begin{bmatrix} x \\ y \end{bmatrix}$$

ADD AN EXTRA COORDINATE TO BUY US SOME REPRESENTATIONAL FLEXIBILITY

HOMOGENEOUS COORDINATES

PROJECTIVE SPACE COORDINATES

$$\tilde{p} \in \mathbb{P}^2 \qquad \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} = \tilde{p}$$

EQUIVALENCE RELATIONSHIP

$\tilde{p}_1 \equiv \tilde{p}_2$ IF AND ONLY IF

$\tilde{p}_1 = k\tilde{p}_2$ FOR SOME $k \neq 0 \in \mathbb{R}$

HOMOGENEOUS EQUIVALENCE

CARTESIAN $\longrightarrow$ HOMOGENEOUS

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \longleftrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in \mathcal{P}^2$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \in \mathbb{R}^2 \rightarrow \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \in \mathcal{P}^2 \equiv \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} \equiv \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$$

$\bar{p}$ AUGMENTED $(\tilde{w}=1)$

CARTESIAN $\longleftarrow$ HOMOGENEOUS

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} \longleftarrow \begin{bmatrix} 12 \\ 9 \\ 3 \end{bmatrix} \equiv \begin{bmatrix} 12/3 \\ 9/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

PERSPECTIVE DIVISION

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \in \mathcal{P}^2 \rightarrow \begin{bmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \end{bmatrix} \in \mathbb{R}^2$$

LINES IN 2D

SLOPE-INTERCEPT FORMULA $y = mx + b$

☹ CAN'T FIND LINE FROM POINTS DIRECTLY ✓

☹ CAN'T DO VERTICAL LINES $m \to \infty$ ✓
$b \to ?$

HOMOGENEOUS GOOD!

IMPLICIT FORMULA: $ax + by + c = 0$

$x = 3$
$\downarrow$
$a = 1$
$b = 0$
$c = -3$

WRITE IN HOMOGENEOUS COORDS:

$$\underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\tilde{l}} \cdot \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\bar{p}} = 0 \qquad \tilde{l} \cdot \bar{p} = 0$$

IF TRUE, FOR ANY NON-ZERO $k \in \mathbb{R}$, ALSO TRUE THAT

$$a(kx) + b(ky) + c(k \cdot 1) = 0$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} kx \\ ky \\ k \end{bmatrix} = 0 \qquad \tilde{l} \cdot \tilde{p} = 0$$

$$\tilde{l}^T \tilde{p} = \tilde{p}^T \tilde{l} = \tilde{p} \cdot \tilde{l}$$

$$(ka)x + (kb)y + kc = 0$$

$$\tilde{l}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \tilde{l}_2 = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix} \qquad \tilde{l}_1 \equiv \tilde{l}_2$$

LIKE POINTS LINES CAN BE HOMOGENEOUSLY EQUIVALENT

FIND LINE THAT PASSES THRU TWO POINTS

GIVEN (x_1, y_1) (x_2, y_2) FIND (a, b, c)

S.T. $ax_1 + by_1 + c = ax_2 + by_2 + c = 0$

$$\begin{aligned} ax_1 + by_1 + c &= 0 \\ ax_2 + by_2 + c &= 0 \\ c &= 1 \end{aligned} \rightarrow \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A \quad x = b$$

$$x = A^{-1} b$$

MIGHT WORK BUT GROSS

- INVERT 3x3,
- ASSUMED $c = 1$

$$\bar{p}_1 = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \qquad \bar{p}_2 = \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$$

FIND $\tilde{\ell} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ S.T. $\tilde{\ell} \cdot \bar{p}_1 = \tilde{\ell} \cdot \bar{p}_2 = 0$

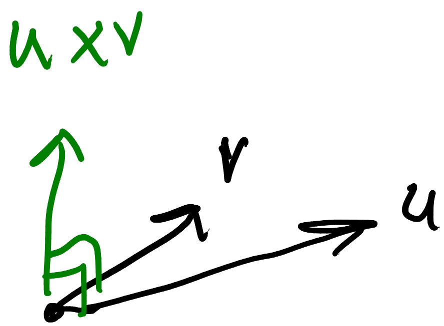
$$\tilde{\ell} \perp \bar{p}_1 \qquad \tilde{\ell} \perp \bar{p}_2$$

CROSS PRODUCT $(\mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}^3$ $\quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

FOR ALL u, V

$$(u \times V) \cdot u = 0$$
$$(u \times V) \cdot V = 0$$

$$u \times V = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$



CHOOSE $\tilde{\ell} = \bar{p}_1 \times \bar{p}_2$

INTERSECTION OF TWO LINES

$$\tilde{l}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \quad \tilde{l}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \quad \text{FIND } \tilde{x}, \tilde{y}, \tilde{w}$$

S.T. $a_1 \tilde{x} + b_1 \tilde{y} + c_1 \tilde{w} = a_2 \tilde{x} + b_2 \tilde{y} + c_2 \tilde{w} = 0$

CHOOSE $\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} = \tilde{l}_1 \times \tilde{l}_2$

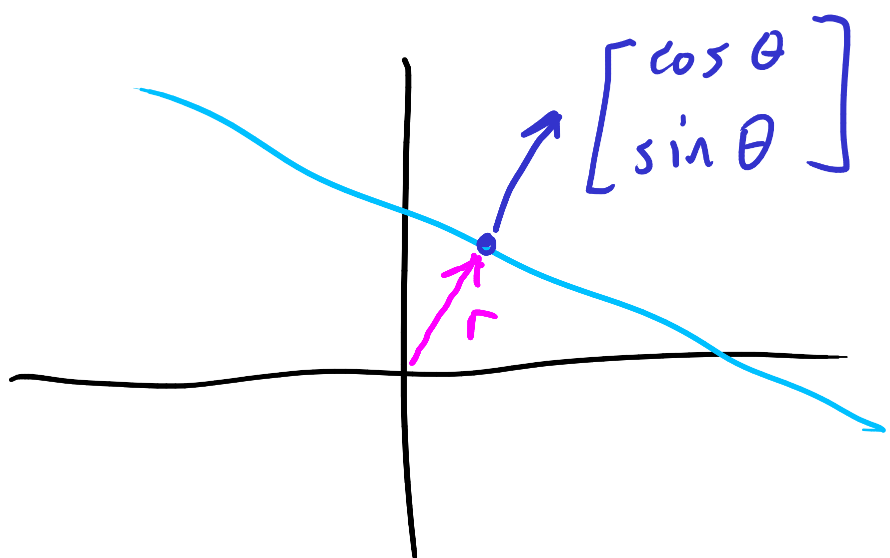
POLAR REPRESENTATION OF LINES

IF WE HAVE $a^2 + b^2 = 1$

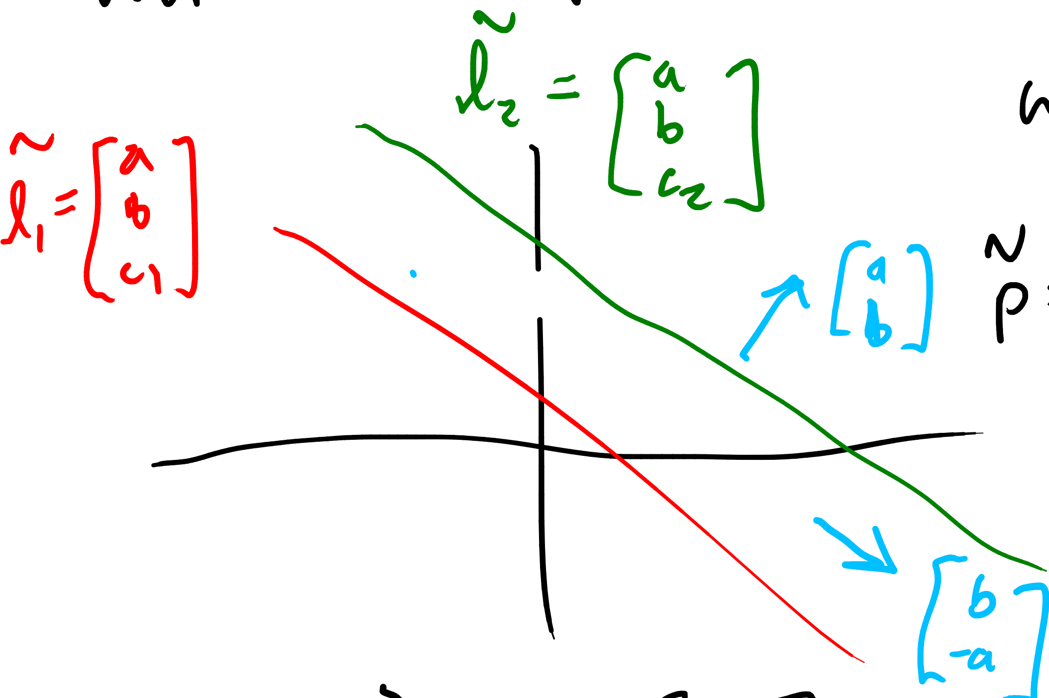
$a = \cos\theta$

$b = \sin\theta$

$c = -r$



WORK AN EXAMPLE – FIND THE INTERSECTION OF 2 LINES



WHAT IS $\tilde{l}_1 \times \tilde{l}_2$?

$$\tilde{p} = \tilde{l}_1 \times \tilde{l}_2 = \begin{bmatrix} bc_2 - bc_1 \\ ac_1 - ac_2 \\ ab - ab \end{bmatrix} = \begin{bmatrix} b(c_2 - c_1) \\ a(c_1 - c_2) \\ 0 \end{bmatrix} = \begin{bmatrix} kb \\ -ka \\ 0 \end{bmatrix}$$

FOR $k = c_2 - c_1$

$$\begin{bmatrix} a & b \\ a & b \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$$

CARTESIAN – IMPOSSIBLE!

$$\tilde{p} \equiv \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix} \equiv \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$$

$\tilde{w}$ IS ZERO!

IDEAL POINT AKA POINT AT INFINITY