WHY HOMOGENEOUS COORDS?

TODAY
- TRANSFORMS of HOMOGENEOUS COORDINATES
- HOMOGRAPHIES

- REPRESENT POINTS & LINES IN SAME FRAMEWORK
- REASON ABOUT POINTS AT ∞
- TODAY: REPRESENT MANY KINDS OF TRANSFORMATIONS AS MATRIX MULTIPLICATION

EXAMPLE: PINHOLE CAMERA PROJECTION

$$P: 3\times 3 \quad \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

focal length (px)

$$P: \mathbb{R}^3 \to \mathbb{P}^2$$

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} fx \\ fy \\ z \end{bmatrix}$$

$$u = \tilde{x}/\tilde{w} = \frac{fx}{z}$$

$$v = \tilde{y}/\tilde{w} = \frac{fy}{z}$$

P IS ONLY DEFINED UP TO SCALE - MULTIPLYING BY A CONSTANT DOESN'T CHANGE THE PROJECTION!

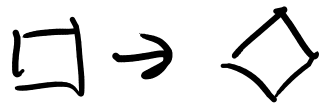
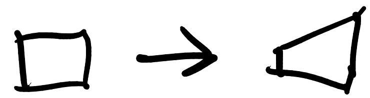
$$\begin{bmatrix} 2f & 0 & 0 \\ 0 & 2f & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 2fX \\ 2fY \\ 2Z \end{bmatrix} \in \mathcal{P}^2$$

$$\Downarrow$$

$$\begin{bmatrix} \frac{fX}{Z} \\ \frac{fY}{Z} \end{bmatrix} \in \mathbb{R}^2$$

HOMOGENEOUS EQUIVALENCE IS DEFINED FOR MATRICES!

HOMOGRAPHIES ENCOMPASS <u>ALL</u> OF THESE

- STRETCH / SHEAR  
- TRANSLATE (MOVE)
- ROTATION 
- PERSPECTIVE / KEYSTONE EFFECT 

REPRESENT AS MATRIX MULTIPLICATION ON HOMOG. COORDS.

EQUIVALENCE! (NOT =)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \equiv \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\tilde{x} = ax + by + c$$
$$\tilde{y} = dx + ey + f$$
$$\tilde{w} = gx + hy + 1$$

PERSPECTIVE DIVISION

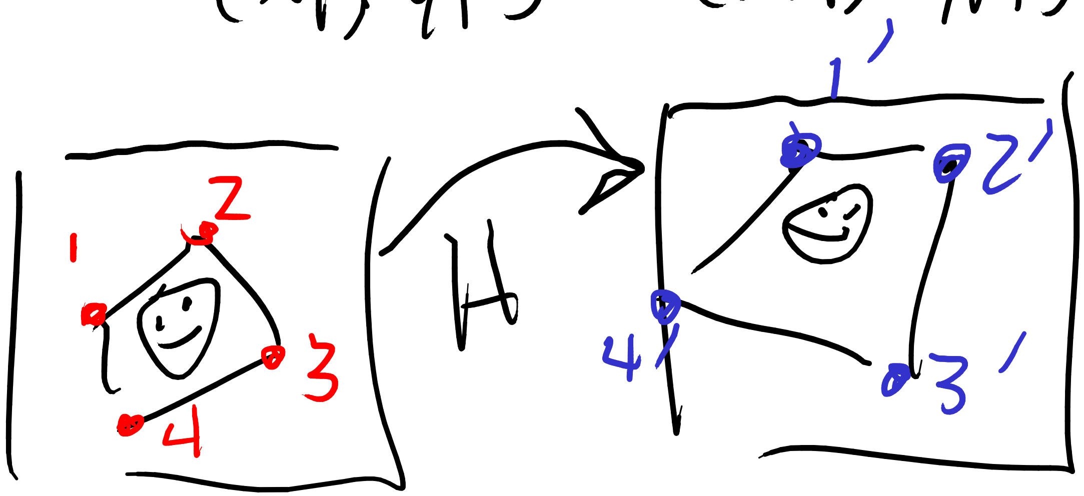
$$x' = \frac{\tilde{x}}{\tilde{w}} = \frac{ax + by + c}{gx + hy + 1}$$

$$y' = \frac{\tilde{y}}{\tilde{w}} = \frac{dx + ey + f}{gx + hy + 1}$$

FIT HOMOGRAPHY FROM DATA

GIVEN $(x_1, y_1) \ldots (x_M, y_M)$ IN IMAGE 1

$(x_1', y_1') \ldots (x_M', y_M')$ IN IMAGE 2



WHAT IS H S.T.

$$\bar{p}_i' \equiv H \bar{p}_i \,?$$

(FOR ALL i)

$$H = \begin{bmatrix} - & h_1^T & - \\ - & h_2^T & - \\ - & h_3^T & - \end{bmatrix} \qquad h_1, h_2, h_3 \in \mathbb{R}^3$$

$$\bar{p}_i = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

GIVEN

$$x_i' = \frac{\bar{p}_i^T h_1}{\bar{p}_i^T h_3} = \frac{\bar{p}_i \cdot h_1}{\bar{p}_i \cdot h_3}$$

GIVEN

UNKNOWN

$$y_i' = \frac{\bar{p}_i^T h_2}{\bar{p}_i^T h_3} = \frac{\bar{p}_i \cdot h_2}{\bar{p}_i \cdot h_3}$$

GOAL: ESTIMATE h_1, h_2, h_3

SO THESE EQUATIONS ARE TRUE FOR ALL i.

$$x_i' \, \bar{p}_i^T h_3 = \bar{p}_i^T h_1 \qquad y_i' \bar{p}_i^T h_3 = \bar{p}_i^T h_2$$

$$\underbrace{\underset{1\times 3}{\bar{p}_i^T} \; \underset{3\times 1}{h_3}}_{1\times 1}$$

$$\bar{p}_i^T h_1 - x_i' \bar{p}_i^T h_3 = 0 \qquad \bar{p}_i^T h_2 - y_i' \bar{p}_i^T h_3 = 0$$

M EQUATIONS IN X   M EQUATIONS IN y

i GOES FROM 1 TO M

$$\begin{array}{cc} & \\ i=1 & \begin{array}{c} x \\ y \end{array} \\ i=2 & \begin{array}{c} x \\ y \end{array} \\ & \\ i=M & \begin{array}{c} x \\ y \end{array} \end{array}
\underbrace{\left[\begin{array}{c|c|c}
\bar{p}_1^T & 0^T \; (1\times 3) & -x_1'\bar{p}_1^T \\
0^T & \bar{p}_1^T & -y_1'\bar{p}_1^T \\
\hline
\bar{p}_2^T & 0^T & -x_2'\bar{p}_2^T \\
0^T & \bar{p}_2^T & -y_2'\bar{p}_2^T \\
\hline
\vdots & \vdots & \vdots \\
\hline
\bar{p}_M^T & 0^T & -x_M'\bar{p}_M^T \\
0^T & \bar{p}_M^T & -y_M'\bar{p}_M^T
\end{array}\right]}_{\underset{3}{\longleftrightarrow}\ \underset{3}{\longleftrightarrow}\ \underset{3}{\longleftrightarrow}}
\begin{bmatrix} \underset{3\times 1}{h_1} \\ \underset{3\times 1}{h_2} \\ \underset{3\times 1}{h_3} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

UNKNOWNS 9×1

$2M \times 9$ A h $2M \times 1$ b

4 POINTS EXACTLY DETERMINES A HOMOGRAPHY

>4 DESIRABLE FOR GOOD FIT

SOLVE THE SYSTEM $Ah = 0$ FOR $A: 2m \times 9$, $h: 9 \times 1$

$\cancel{h = A^{-1} 0}$ A IS NOT SQUARE

TRY ORDINARY LEAST SQUARES (O.L.S.)

$$h = (A^T A)^{-1} A^T \vec{0} \Rightarrow h = 0$$

JUST GIVES TRIVIAL SOLUTION! NEED TO FIND ANOTHER WAY

RESTRICT TO $\|h\|^2 = 1$ $\sum_{i,j} h_{ij}^2 = 1$

MINIMIZE $\|Ah - \vec{0}\|^2 = \|Ah\|^2$ S.T. $\|h\| = 1$

HOMOGENEOUS LEAST SQUARES PROBLEM (H.L.S.)

SIMILAR TO PCA $\rightarrow$ CHOOSE UNIT VECTOR x TO MAXIMIZE $\|Ax\|^2$

PCA SOLUTION: CHOOSE E.VEC. OF $(A^T A)$ WITH BIGGEST E.VAL.

H.L.S.: CHOOSE E.VEC. OF $(A^T A)$ WITH SMALLEST E.VALUE