# A PRACTICAL (RE-)INTRODUCTION TO LINEAR ALGEBRA

MATT ZUCKER 9/2014

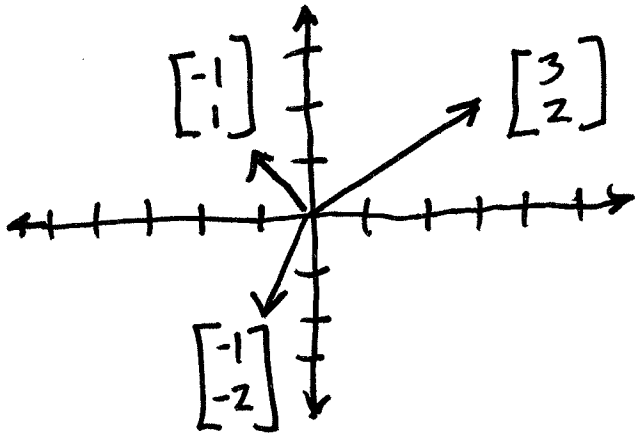
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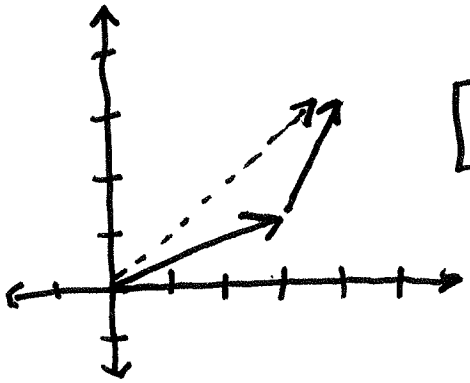
# A PRACTICAL (RE-) INTRODUCTION TO LINEAR ALGEBRA

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LET'S START WITH VECTORS. IN ENGINEERING, WE ARE USUALLY CONCERNED WITH REAL-VALUED VECTORS IN 2D OR 3D. IN SHORT, VECTORS ARE QUANTITIES WITH BOTH DIRECTION AND MAGNITUDE. HERE ARE SOME IN 2D:

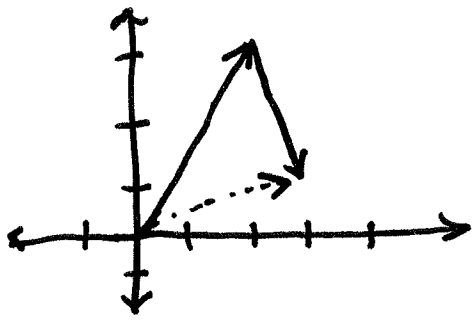



VECTORS CAN BE ADDED:



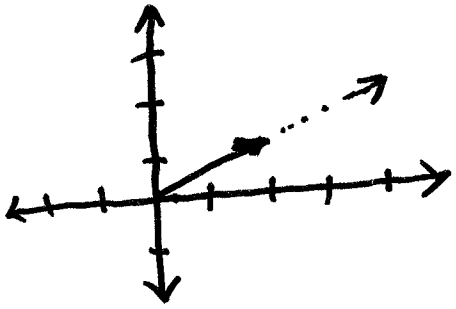
$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

... SUBTRACTED:



$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

... AND MULTIPLIED BY SCALARS:



$$2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

SCALAR ↔ "PLAIN OLD NUMBER"

TOGETHER, THESE OPERATIONS ALLOW US TO CONSTRUCT LINEAR COMBINATIONS – WEIGHTED SUMS OF VECTORS OF THE FORM

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

SCALARS

VECTORS

VECTOR RESULTING FROM LINEAR COMBINATION

VECTORS LIVE IN AN UNDERLYING VECTOR SPACE – FOR US, TYPICALLY 2D OR 3D. FORMALLY, THAT'S $\mathbb{R}^2$ OR $\mathbb{R}^3$.

$\mathbb{R}^2$ ↔ "space of real numbers of dimension 2"

NOTE THAT ADDITION, SUBTRACTION, AND MULTIPLICATION BY SCALARS DON'T CHANGE THE UNDERLYING VECTOR SPACE!

IN CONTRAST TO THE CONCEPT OF LINEAR COMBINATIONS, WE CAN TALK ABOUT LINEAR INDEPENDENCE. A SET OF $n$ VECTORS $\{v_1, v_2, \ldots, v_n\}$ IS DEFINED TO BE <u>LINEARLY INDEPENDENT</u> IF IT IS IMPOSSIBLE TO WRITE ANY VECTOR IN THE SET AS A LINEAR COMBINATION OF THE REMAINING ONES.

<u>EXAMPLE:</u> THE SETS

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

ARE LINEARLY INDEPENDENT. HOWEVER,

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix} \right\}$$

ARE NOT.

A LINEARLY INDEPENDENT SET OF VECTORS IN $\mathbb{R}^n$ OF SIZE $n$ (THAT IS, THE SIZE OF THE SET IS EQUAL TO THE DIMENSION OF THE SPACE) IS SAID TO BE A <u>BASIS</u> FOR $\mathbb{R}^n$, AND <u>ANY</u> VECTOR IN THE SPACE CAN BE EXPRESSED AS SOME LINEAR COMBINATION OF THE VECTORS IN THE SET.

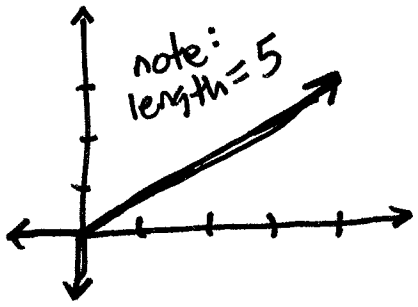
THERE'S ONE MORE OPERATION WE CARE ABOUT DEFINED ON VECTORS - THE DOT PRODUCT. THE DOT PRODUCT OF TWO VECTORS IN THE SAME SPACE IS DEFINED AS

$$u \cdot v = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

WE CAN NOW ALSO FORMALLY DEFINE THE MAGNITUDE OF A VECTOR AS

$$\|u\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

(THIS IS BASICALLY JUST A GENERALIZATION OF THE PYTHAGOREAN THEOREM:)

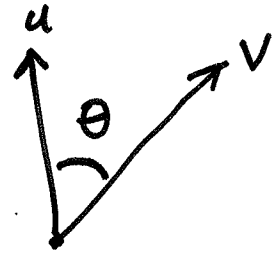



$$\left\| \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\| = \sqrt{4^2 + 3^2} = 5$$

THE DOT PRODUCT TELLS US INFORMATION ABOUT THE ANGLE BETWEEN TWO VECTORS. IN FACT,

$$u \cdot v = \cos\theta \, \|u\| \, \|v\|$$

NOTE: MAGNITUDE ALWAYS POSITIVE



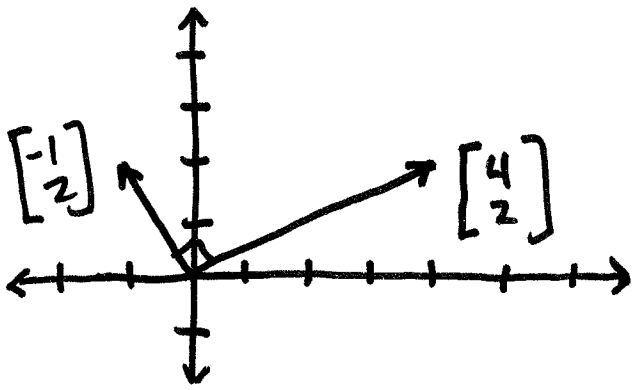

HENCE,

$u \cdot v > 0 \iff \theta < 90^\circ$ (acute)

$u \cdot v = 0 \iff \theta = 90^\circ$ ($u \perp v$)

$u \cdot v < 0 \iff \theta > 90^\circ$ (obtuse)

ILLUSTRATION OF $\theta = 90°$ CASE:




$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4+4 = 0$$

BESIDES VECTORS, WE ARE ALSO INTERESTED IN MATRICES, WHICH ACT ON VECTORS TO ALTER THEIR DIRECTION AND MAGNITUDE – AND POSSIBLY EVEN THEIR DIMENSION! THIS MAPPING CAN BE WRITTEN AS A MATRIX-VECTOR PRODUCT, LIKE THIS:

$$u = Av$$

$u$: vector in $\mathbb{R}^m$ — $A$: $m \times n$ Matrix — $v$: vector in $\mathbb{R}^n$

SO WHEREAS VECTORS HAVE A SINGLE DIMENSION, MATRICES HAVE TWO – THE DIMENIONS OF BOTH THE OUTPUT AND THE INPUT SPACES. WE WRITE THEM AS RECTANGULAR ARRAYS OF NUMBERS:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$A$: $m \times n$ Matrix

OPERATIONALLY, WE DEFINE THIS MATRIX-VECTOR PRODUCT AS

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n \\ a_{21}u_1 + a_{22}u_2 + \ldots + a_{2n}u_n \\ \vdots \\ a_{m1}u_1 + a_{m2}u_2 + \ldots + a_{mn}u_n \end{bmatrix}$$

$m \times n$ matrix; vector in $\mathbb{R}^n$; vector in $\mathbb{R}^m$

MATRICES ACT AS <u>LINEAR TRANSFORMATIONS</u> ON VECTORS, WHICH MEANS THAT FOR ANY MATRIX $A$, ANY VECTORS $u$ & $v$, AND ANY SCALARS $\alpha$ & $\beta$,

$$A(\alpha u + \beta v) = \alpha Au + \beta Av$$

THIS IS IMPORTANT ENOUGH TO WRITE AS TWO SEPARATE PROPERTIES:

$$A(u+v) = Au + Av \quad \text{[distributes over addition]}$$

$$A(\alpha u) = \alpha Au \quad \text{[commutes with scalar product]}$$

WE CAN ALSO DEFINE THE <u>TRANSPOSE</u> OPERATION ON MATRICES AS FLIPPING ABOUT THE DIAGONAL — OR, EQUIVALENTLY, EXCHANGING ROWS AND COLUMNS.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

flip on this line

IF WE SQUINT OUR EYES, WE CAN PRETEND A VECTOR IN $\mathbb{R}^n$ IS BASICALLY AN $n \times 1$ MATRIX... THIS ALLOWS US TO REDEFINE THE DOT PRODUCT AS A MATRIX-VECTOR PRODUCT USING THE TRANSPOSE:

$$u \cdot v = u^T v = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \ldots + u_n v_n$$

THERE ARE TWO OTHER USEFUL WAYS TO CONCEPTUALIZE MATRIX-VECTOR MULTIPLICATION. THE FIRST ONE IS...

## COLUMN-WISE

$$v = Au = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1 c_1 + u_2 c_2 + \ldots + u_n c_n$$

- $v$: vector in $\mathbb{R}^m$
- $A$: $m \times n$ matrix
- $u$: vector in $\mathbb{R}^n$
- $c_1, c_2, \ldots, c_n$: columns of $A$ as vectors in $\mathbb{R}^m$
- $u_1, u_2, \ldots, u_n$: elements of $u$

IN THIS SENSE, WE SEE THE ELEMENTS OF $u$ SERVING AS WEIGHTS ON THE COLUMNS OF $A$ IN A LINEAR COMBINATION.

## ROW-WISE

$$v = Au = \begin{bmatrix} — & r_1^T & — \\ — & r_2^T & — \\ & \vdots & \\ — & r_m^T & — \end{bmatrix} u = \begin{bmatrix} r_1^T u \\ r_2^T u \\ \vdots \\ r_m^T u \end{bmatrix} = \begin{bmatrix} r_1 \cdot u \\ r_2 \cdot u \\ \vdots \\ r_m \cdot u \end{bmatrix}$$

rows of $A$ as transposed vectors in $\mathbb{R}^n$.

IN THIS SENSE, WE SEE THE ELEMENTS OF $v$ ARE THE DOT PRODUCT OF EACH ROW OF $A$ WITH $u$.

LET'S DISCUSS SOME USEFUL IDENTITIES. THESE ARE OPERATIONS WHICH LEAVE THEIR ARGUMENTS UNCHANGED. THE IDENTITY WITH RESPECT TO VECTOR ADDITION OR SUBTRACTION IS THE NULL VECTOR, WHICH HAS ALL ZERO ELEMENTS:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

THE IDENTITY WITH RESPECT TO MULTIPLYING MATRIX-VECTOR PRODUCTS IS THE IDENTITY MATRIX, WHICH HAS ONES ON THE DIAGONAL AND ZEROS EVERYWHERE ELSE:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

NOTE: WE WRITE THE IDENTITY MATRIX AS $I$ (ITS DIMENSION IS INFERRED FROM CONTEXT.)

WE CAN ALSO DISCUSS THE NULL SPACE OF A MATRIX, DEFINED AS THE SET OF ALL VECTORS $u$ SUCH THAT $Au$ = THE NULL VECTOR. FOR EXAMPLE, THE NULL SPACE OF THE MATRIX

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

IS ANY SCALAR MULTIPLE OF THE VECTOR $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

TRY IT!

WE CAN COMPOSE A SEQUENCE OF LINEAR TRANSFORMATIONS BY DEFINING THE MATRIX-MATRIX PRODUCT

$$v = A(Bu) = (AB)u$$

$v$: vector in $\mathbb{R}^k$; $A$: $k \times m$ matrix; $B$: $m \times n$ matrix; $u$: vector in $\mathbb{R}^n$; $AB$: $k \times n$ matrix product

WE CAN DEFINE IT IN TERMS OF THE COLUMNS OF $B$ BY:

$$AB = A\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_n \\ | & | & & | \end{bmatrix}$$

columns of $B$ as vectors in $\mathbb{R}^m$

NOTE IN GENERAL $AB \neq BA$. ORDER MATTERS! *
MATRIX-MATRIX PRODUCTS INTERACT WITH THE TRANSPOSE OPERATION ACCORDING TO:

$$(AB)^T = B^T A^T$$

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* EXCEPT FOR THE DOT PRODUCT, WHERE $u \cdot v = v \cdot u = u^T v = v^T u$.

FINALLY LET US DEFINE THE INVERSE OF A MATRIX. AN $n \times n$ (SQUARE) MATRIX $A$ MAY HAVE AN INVERSE $A^{-1}$ WHICH SATISFIES

$$AA^{-1} = A^{-1}A = I \leftarrow \text{identity matrix}$$

IF $A^{-1}$ EXISTS, IT IS UNIQUE. OBVIOUSLY, MATRIX INVERSES COME IN HANDY FOR SOLVING EQUATIONS LIKE

$$Ax = b$$

FOR THE VECTOR $x$. THE SOLUTION IS OBTAINED BY LEFT-MULTIPLYING BOTH SIDES BY $A^{-1}$:

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

A HANDY FORMULA FOR INVERTING A $2 \times 2$ MATRIX IS

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

THE TERM $(ad-bc)$ IS CALLED THE DETERMINANT OF A. CLEARLY, IF IT IS ZERO, THE INVERSE IS UNDEFINED.

DETERMINANTS CAN BE COMPUTED FOR LARGER MATRICES TOO, BUT IT'S A BIT OF A PAIN, FRANKLY.*

TECHNICALLY, $A^{-1}$ EXISTS IF ANY OF THE FOLLOWING ARE TRUE:

- THE ROWS OF A ARE LINEARLY INDEPENDENT.
- THE COLUMNS OF A ARE LINEARLY INDEPENDENT.
- THE NULL SPACE OF A IS JUST THE NULL VECTOR.
- THE DETERMINANT OF A IS NON-ZERO.

IF A AND B ARE SQUARE MATRICES AND $A^{-1}$ AND $B^{-1}$ EXIST THEN $(AB)^{-1} = B^{-1}A^{-1}$.

THERE'S A TON MORE TO TALK ABOUT (LIKE EIGENVECTORS AND EIGENVALUES), BUT THIS SHOULD BE SUFFICIENT TO GET YOU RUNNING IN AN ENGINEERING SETTING. GO FLIP THROUGH A LINEAR ALGEBRA TEXTBOOK FOR MORE INFO!

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* SEE SHELDON AXLER, DOWN WITH DETERMINANTS, (1995).**

** THIS IS ACTUALLY A REAL PAPER. SERIOUS.